Quant Puzzles 8 - Solution Spaces

Problem 1 - Number of Solutions [Medium]

how many distinct solutions does the following equation have?

$$x_1 + x_2 + x_3 + x_4 = 100, x_i \in \{0, 1, 2..., 100\}$$

solution: 17,6851

We will explain this problem through an analogy. Suppose we have a certain number of sticks, and dividers between those sticks. Let the desired result of our equation symbolize the number of sticks we have. We want to know how many ways we can fit these sticks between our dividers.

- Let a “|” symbol denote a stick
- Let a “+” symbol denote a divider

Then the number of solutions to the equation is equal to the number of ways to arrange 100 sticks and 3 dividers. Take some time to convince yourself that these two problems are equivalent.

Now recall a combinatorial property that $\binom{n}{k} = \frac{n!}{(n-k)!k!} = \binom{n}{(n-k)}$

We have a total of 100 sticks and 3 dividers and the total number of items amount to $n = 103$. Suppose we have all of our 100 sticks laid out and need to enumerate the ways to assign 3 dividers, or that we have 3 dividers laid out and need to determine where to place the sticks. Both problems are equivalent and we get the following number of solutions to the equation:

$$\binom{103}{3} = \binom{103}{100} = 17,6851$$

Problem 2 - Number of Solutions Restricted [Hard]

How many distinct solutions does the following equation have?

$$x_1 + x_2 + x_3 + x_4 = 100, x_1 \in \{1, 2, 3, ...\}, x_2 \in \{2, 3, 4, ...\}, x_3, x_4 \in \{0, 1, 2, 3, ...\}$$

This time we have placed restrictions on the domains of the variables. This simplifies our problem. Due to the restrictions placed on x_1 and x_2, we have already assigned a minimum of $(1 + 2 = 3)$ sticks. We have 97 sticks left to assign along with 3 dividers. Therefore, the total number of items to assign amounts to $97 + 3 = 100$. Following the pattern from problem 1, we get the following solution.

$$\binom{100}{3} = \binom{100}{97} = 161700$$