Quant Puzzles 3: Infinities and Probabilities

Question 1:

The geometric distribution gives us the discrete probability of an event occurring on the nth trial.

$$P(X=n)=(1-p)^{n-1}p,$$

Derive the mean and variance of the geometric distribution. The final answer should be an expression in terms of **p** (for both the mean as well as the variance).

$$E[X]=\sum_{i=1}^{\infty} i(1-p)^{i-1}p = p\sum_{i=1}^{\infty} i(1-p)^{n-1} = p\cdot[1+2(1-p)+3(1-p)^2+4(1-p)^3+...]$$
$$pE[X]=E[X]-(1-p)E[X]=p[1+2(1-p)+3(1-p)^2+4(1-p)^3+...]-p[(1-p)+2(1-p)^2+3(1-p)^3+$$
$$=p\cdot[1+(1-p)+(1-p)^2+(1-p)^3+...]$$
$$=p\cdot\frac{1}{1-(1-p)}=1$$
$$\therefore pE[x]=1 \implies E[x]=\frac{1}{p}\square$$

variance:

$$VAR[X]=E[X^2]-E[X]^2=E[X^2]-\frac{1}{p^2}=p\sum_{i=1}^{\infty} i^2(1-p)^{i-1}-\frac{1}{p^2}$$
$$E[X^2]=p\sum_{i=1}^{\infty} i^2(1-p)^{i-1}=p[1+4(1-p)+9(1-p)^2...+i_{max}^2(1-p)^{i_{max}-1}]$$
$$(1-p)E[X^2]=p[(1-p)+4(1-p)^2+9(1-p)^2+...+i_{max}^2(1-p)^{i_{max}-1}]$$
$$\implies E[X^2]-(1-p)E[X^2]=pE[X^2]$$
$$=p[1+3(1-p)+5(1-p)^2+7(1-p)^3+...]$$
$$\implies (1-p)pE[X^2]=p[(1-p)+3(1-p)^2+5(1-p)^2+...]$$
$$\therefore pE[X^2]-(1-p)pE^2]$$
$$=p^2E[X^2]=p[1+2(1-p)+2(1-p)^2+2(1-p)^3+2(1-p)^4...]=p[1+2\sum_{i=1}^{\infty}(1-p)^i]=p[1+2(\frac{1-p}{p})]$$
$$\therefore E[X^2]=\frac{1}{p}[1+2(\frac{1-p}{p})]=\frac{2-p}{p^2}$$
$$\implies VAR[X]=E[X^2]-E[X]^2=\frac{2-p}{p^2}-\frac{1}{p^2}=\frac{1-p}{p^2}\square$$

Question 2:

We have some discrete random variable **X,** a natural number that takes the following values with some probability:

$$Range(X)=\{1,....,\infty\}$$

Prove the following:

$$E[X] = \sum_{x=1}^{\infty} P(X \geq x)$$

$$\begin{aligned} E[X] = \sum_{x=1}^{\infty} xP(X = x) = P(X = 1) + 2P(X = 2) + 3P(X = 3) + ... \\ = [P(X = 1) + P(X = 2) + P(X = 3) + ...] + [P(X = 2) + P(X = 3) + P(X = 4) + ...] \\ +[P(X = 3) + P(X = 4) + P(X = 5) + ...] + ... \\ = \sum_{x=1}^{\infty} P(X = x) + \sum_{x=2}^{\infty} P(X = x) + \sum_{x=3}^{\infty} P(X = x) + ... \\ = P(X \geq 1) + P(X \geq 2) + P(X \geq 3) + ... \\ = \sum_{x=1}^{\infty} P(X \geq x) \square \end{aligned}$$

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