# NT Quant Quant Puzzles 4 : Recursion

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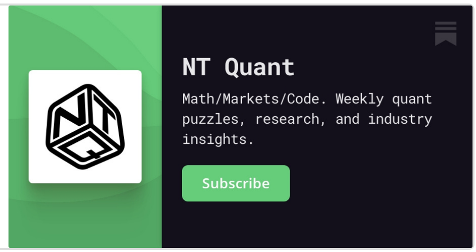


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## Problem 1: Down to Zero [Easy]

Suppose a random process starting at node $n$ moves to node $n+1$ with probability $p$, and node $n-1$ with probability **q**. Suppose also that **q > p.** The process ends if it arrives at node $0$. What is the expected number of steps to reach node **0** in terms of **p**, **q**, and **n**?

**Solution:**

This question is a simple application of the linearity of expectation.

$$EX = p(1) + q(-1) = p - q < 0$$

Every iteration, we move **|p - q|** steps in the negative direction. So the expected number of steps given that we start at node n is:

$$s = \frac{n}{|p-q|}$$

# Problem 2: Before Reaching 0 [Medium]

Suppose a random process starting at node **n** moves to node **n+1** with probability **1/3**, and node **n-1** with probability **2/3**. The process ends if it arrives at node $0$.

What is the probability of reaching node 5 before node 0 if you start at node 3?

**Solution:**

We form a system of equations to express this problem. Let $s_i$ be the probability of reaching state 5 before reaching state 0 from state $i$.

$$s_0 = 0$$
$$s_1 = \frac{1}{3}s_2 + \frac{2}{3}s_0 = \frac{1}{3}s_2$$
$$s_2 = \frac{1}{3}s_3 + \frac{2}{3}s_1$$
$$s_3 = \frac{1}{3}s_4 + \frac{2}{3}s_2$$
$$s_4 = \frac{1}{3}s_5 + \frac{2}{3}s_3$$
$$s_5 = 1$$

Solving the system of equations, we get that

$$s_3 = \frac{7}{31}$$

# Problem 3: Before Reaching 0, General Case [Hard]

Suppose we have a set of states

$$\{0, 1, ...N\}$$

the probability of going from some state **n** to **n+1** is **p**, the probability of going from state **n** to **n-1** is **q = (1-p).** We want to find the probability of reaching state **N** before reaching state **0** given that we are located at any starting state such that

$$n \neq 0, n \neq N$$

Find a matrix formulation of this problem, and a matrix expression for the solution.

**Solution:**

$$s_0 = 0$$
$$s_1 = ps_2$$
$$s_2 = ps_3 + qs_1$$

$$s_3 = ps_4 + qs_2$$
$$...$$
$$s_n = ps_{n+1} + qs_{n-1}, \forall 0 < n < N$$

This is a simple linear system. We can build a matrix to express it.

create an $(N+1) \times (N+1)$ matrix where each column represents a state and each row represents an equation.

Let $A$ be a $(N+1) \times (N+1)$ matrix where

$$A_{11} = 1, A_{1,j>1} = 0$$

$$A_{N+1,N+1} = 1, A_{N+1,j<N+1} = 0$$

$$A_{i,i-1} = q, A_{ii} = -1, A_{i,i+1} = p, 1 < i < N+1$$

Express the linear system as

$$A\vec{s} = \vec{b}$$

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & ... & 0 \\
0 & -1 & p & 0 & 0 & 0 & ... & 0 \\
0 & q & -1 & p & 0 & 0 & ... & 0 \\
0 & 0 & q & -1 & p & 0 & ... & 0 \\
0 & 0 & 0 & q & -1 & p & ... & 0 \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & 0 & 0 & ... & 1
\end{bmatrix}
\begin{bmatrix}
s_0 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ . \\ . \\ . \\ s_N
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ . \\ . \\ 0 \\ 1
\end{bmatrix}
$$

then,

$$\vec{s} = A^{-1}\vec{b}$$